Exercise: Metropolis–Hastings samplers for bivariate densities

We will consider three different bivariate target densities for $\boldsymbol{x} = (x_1, x_2)^t$:

1. Standard Gaussian distribution with correlation:
$$\pi(\boldsymbol{x}) = \frac{1}{2\pi|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{x}^t\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\right),$$
where $\boldsymbol{\Sigma}$ has 1 on the diagonal and $\rho = 0.9$ on the off-diagonal.

2. A banana-shaped density:
$$\pi(\boldsymbol{x}) = \sum_{i=1}^{5} w_i \frac{1}{2\pi|\boldsymbol{\Sigma}_i|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)\right),$$
which is a mixture of Gaussian densities where weights are $w_1 = 0.125$, $w_2 = 0.25$, $w_3 = 0.25$, $w_4 = 0.25$ and $w_5 = 0.125$, means are $\boldsymbol{\mu}_1 = (-2, 1)^t$, $\boldsymbol{\mu}_2 = (-1, 0)^t$, $\boldsymbol{\mu}_3 = (0, -1)^t$, $\boldsymbol{\mu}_4 = (1, 0)^t$ and $\boldsymbol{\mu}_5 = (2, 1)^t$, and covariance matrices $\boldsymbol{\Sigma}_i$ all have variance 0.5^2 and correlation $\rho_1 = -0.9$, $\rho_2 = -0.5$, $\rho_3 = 0$, $\rho_4 = 0.5$ and $\rho_5 = 0.9$.

3. A volcano-shaped density:
$$\pi(\boldsymbol{x}) \propto \frac{1}{2\pi} \exp\left(-\frac{1}{2}\boldsymbol{x}^t\boldsymbol{x}\right)(\boldsymbol{x}^t\boldsymbol{x} + 0.25),$$

We will explore each one with random walk Metropolis–Hastings (MH) algorithms, Langevin MH and Hamiltonian MH.

1 Plotting

Visualize the three densities on a grid covering $[-3, 3] \times [-3, 3]$. The grid spacing could be 0.1, which gives $61 \times 61 =$ grid cells. Note that the volcano-density in 3. is not normalized, but the relative levels are still representative.

2 Random walk MH

- Implement a random walk MH sampler for the Gaussian density in 1. above. Try tuning parameter $\sigma = 0.5$ in the random walk proposal, and then experiment with a few others. Keep track of the mean acceptance rate. Approximate the integrated autocorrelation of the resulting Markov chain, and use this along with trace plots of x_1 and x_2 to evaluate the suitability of the different levels of the tuning parameter.

- Implement a random walk MH sampler for the banana-shaped density in 2. above. Repeat the same algorithm evaluation as for the Gaussian density above.

- Implement a random walk MH sampler for the volcano-shaped density in 3. above. Repeat the same algorithm evaluation as above.

3 Langevin MH

- Implement a Langevin MH sampler for the Gaussian density in 1. above. Try tuning parameter $\sigma = 0.5$ in the Langevin proposal, and then experiment with a few others. Keep track of the mean acceptance rate. Approximate the integrated autocorrelation of the resulting Markov chain, and use this along with trace plots of x_1 and x_2 to evaluate the suitability of the different levels of the tuning parameter. Compare also with the Random Walk MH in the previous section. Keep in mind that each iteration of the Langevin MH sampler relies on the evaluations of both target and derivative.

- Implement a Langevin MH sampler for the banana-shaped density in 2. above. Repeat the same algorithm evaluation as for the Gaussian density above.

- Implement a Langevin MH sampler for the volcano-shaped density in 3. above. Repeat the same algorithm evaluation as above.

4 Hamiltonian MH

- Implement a Hamiltonian MH sampler for the Gaussian density in 1. above. Set the momentum proposal to $\boldsymbol{z} \sim N(0, \boldsymbol{I})$. Try tuning parameter $\epsilon = 0.1$ in the leap-frog scheme for $T = 5$ steps, and then experiment with a few other settings. Keep track of the mean acceptance rate. Approximate the integrated autocorrelation of the resulting Markov chain, and use this along with trace plots of x_1 and x_2 to evaluate the suitability of the different levels of the tuning parameter. Compare also with the Random Walk and Langevin MH in the previous sections. Keep in mind that each iteration of the Hamiltonian MH sampler requires several evaluations of both target and derivative in the leap-frog calculation.

- Implement a Hamiltonian MH sampler for the banana-shaped density in 2. above. Repeat the same algorithm evaluation as for the Gaussian density above.

- Implement a Hamiltonian MH sampler for the volcano-shaped density in 3. above. Repeat the same algorithm evaluation as above.