Uncertainty and statistics
Part I

Håkon Tjelmeland

CGF and Department of Mathematical Sciences
Norwegian University of Science and Technology

Friday October 22nd 2021

Plan for the course

- ⋆ Part I: Introduction, stochastic variables and the effect of conditioning
- ⋆ Part II: Modelling of dependence, conditional independence
- ⋆ Part III: Bayesian inversion, prior and posterior distribution
- ⋆ Part IV: Spatial model for categorical variables, Markov chain Monte Carlo
- ⋆ Part V: Dynamic state space models, Kalman and ensemble Kalman filters

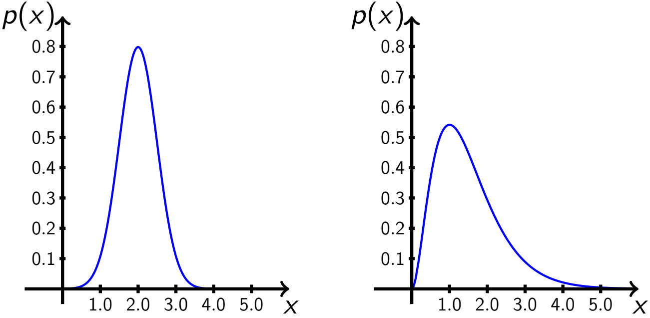
- ⋆ Part I, II and III: Focus on models and intuition
  - want to understand the results from the models
  - mostly small toy examples
- ⋆ Part IV and V: Focus on algorithms and larger examples
  - what is computationally feasible?
  - larger examples

Plan today

- ⋆ One stochastic variable
  - distribution, $p(x)$
  - mean and variance (standard deviation)
  - estimation from Monte Carlo samples
- ⋆ Two stochastic variables
  - joint distribution, $p(x_1, x_2)$
  - correlation
  - marginal and conditional distributions
  - the effect of conditioning
  - how to specify a model, $p(x_1, x_2)$
  - some simple examples

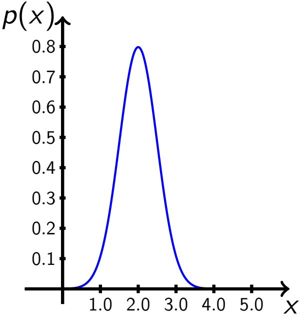
One stochastic variable

- ⋆ Continuous stochastic variable, X
  - amount of gas in a reservoir
  - porosity
  - your favourite uncertain quantity

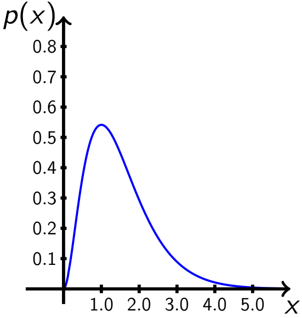


One stochastic variable

- Continuous stochastic variable, X
  - amount of gas in a reservoir
  - porosity
  - your favourite uncertain quantity



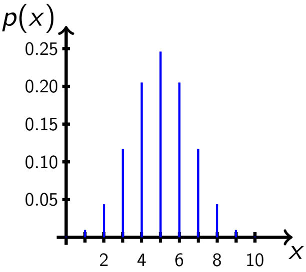
$$p(x) = \sqrt{\frac{2}{\pi}} e^{-2(x-2)^2}$$

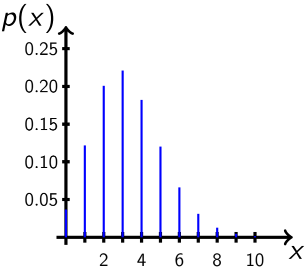


$$p(x) = \frac{e^{-2x}}{\sqrt{2x}}, x > 0$$

One stochastic variable

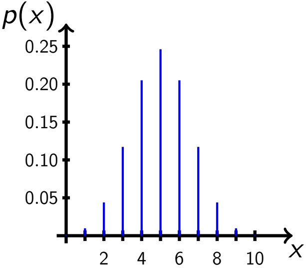
⋆ Discrete stochastic variable, X

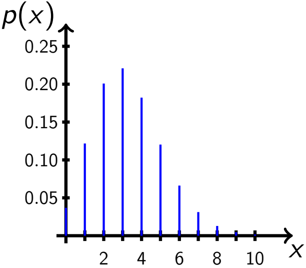




One stochastic variable

⋆ Discrete stochastic variable, X

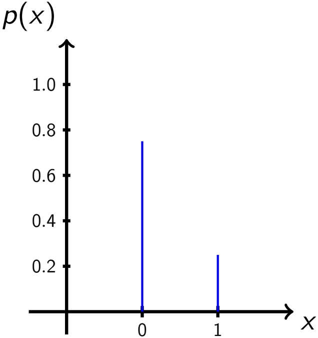




$p(x) = \binom{10}{x} 0.5^{10}, x = 0, 1, \ldots, 10$ $\quad$ $p(x) = \frac{3.3^x}{x!} e^{-3.3}, x = 0, 1, \ldots$

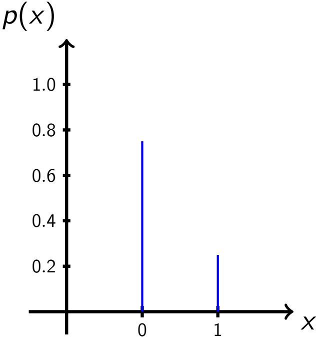
One stochastic variable

- Binary stochastic variable, X
  - presence/not presence of hydrocarbons
  - CO2 leakage/not CO2 leakage



One stochastic variable

- ⋆ Binary stochastic variable, X
  - presence/not presence of hydrocarbons
  - CO_2 leakage/not CO_2 leakage



- ⋆ Categorical stochastic variable, X
  - shale, oil filled sandstone, brine sandstone
  - red, blue, green, yellow

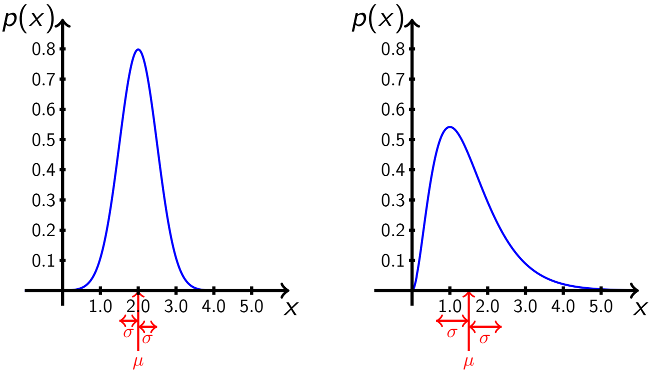
Mean and variance

- ⋆ Mean:

$$\mu = \mathrm{E}[X] = \int_{-\infty}^{\infty} x p(x) dx$$

- ⋆ Variance and standard deviation:

$$\sigma^2 = \mathrm{Var}[X] = \mathrm{E}[(X-\mu)^2] \quad \text{and} \quad \sigma = \mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]}$$

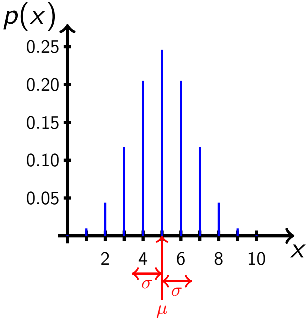


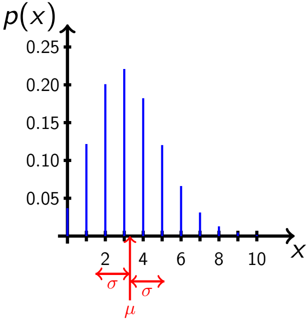
Mean and variance

- ⋆ Mean: $$\mu = \mathrm{E}[X] = \sum_x x p(x)$$

- ⋆ Variance and standard deviation:

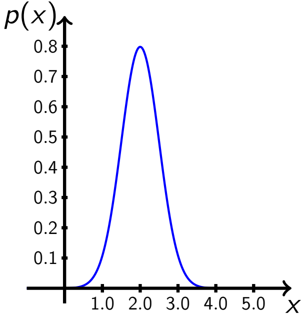
$$\sigma^2 = \mathrm{Var}[X] = \mathrm{E}[(X-\mu)^2] \quad \text{and} \quad \sigma = \mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]}$$

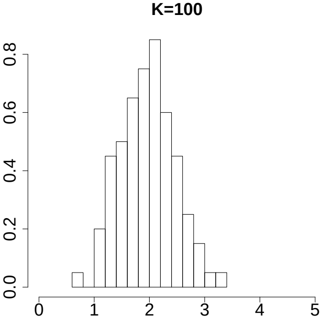




Monte Carlo simulation

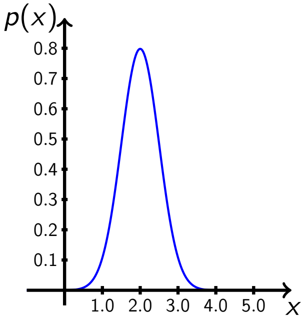
- ⋆ If you don't have a (simple) formula for $p(x)$,
  - how to "plot" $p(x)$?
  - how to "compute" $\mathrm{E}[X]$ and $\mathrm{Var}[X]$?
- ⋆ Often possible to generate Monte Carlo samples from $p(x)$: $x_1, x_2, \ldots, x_K$

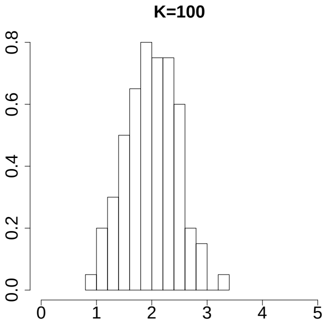




Monte Carlo simulation

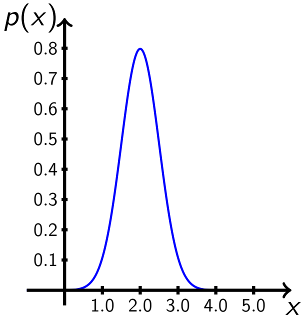
- ⋆ If you don't have a (simple) formula for $p(x)$,
  - how to "plot" $p(x)$?
  - how to "compute" $E[X]$ and $Var[X]$?
- ⋆ Often possible to generate Monte Carlo samples from $p(x)$: $x_1, x_2, \ldots, x_K$

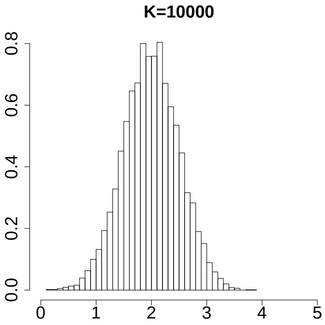




Monte Carlo simulation

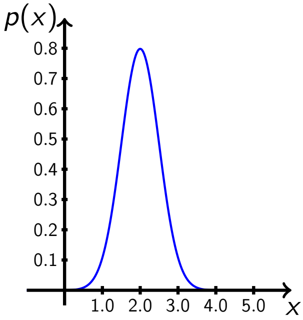
- ⋆ If you don't have a (simple) formula for $p(x)$,
  - how to "plot" $p(x)$?
  - how to "compute" $E[X]$ and $Var[X]$?
- ⋆ Often possible to generate Monte Carlo samples from $p(x)$: $x_1, x_2, \ldots, x_K$

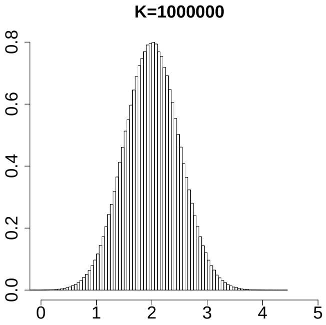




Monte Carlo simulation

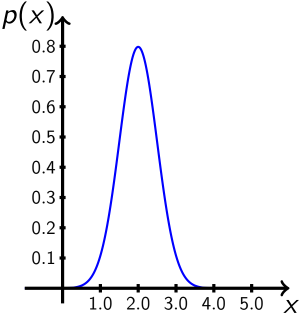
- ⋆ If you don't have a (simple) formula for $p(x)$,
  - how to "plot" $p(x)$?
  - how to "compute" $E[X]$ and $Var[X]$?
- ⋆ Often possible to generate Monte Carlo samples from $p(x)$: $x_1, x_2, \ldots, x_K$

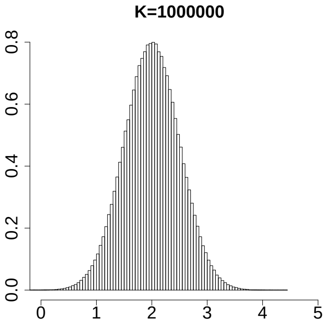




Monte Carlo simulation

- ⋆ If you don't have a (simple) formula for $p(x)$,
  - how to "plot" $p(x)$?
  - how to "compute" $\mathrm{E}[X]$ and $\mathrm{Var}[X]$?
- ⋆ Often possible to generate Monte Carlo samples from $p(x)$: $x_1, x_2, \ldots, x_K$





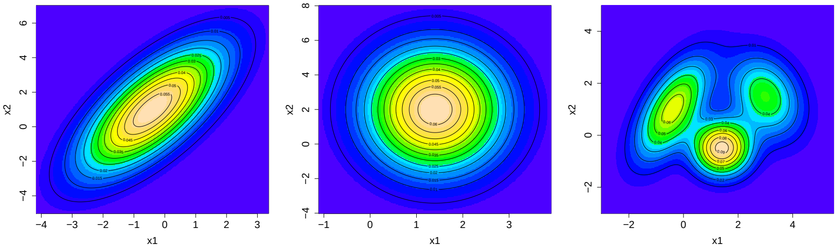
$$\widehat{\mu} = \bar{x} = \frac{1}{K}\sum_{i=1}^{K} x_i \qquad \text{and} \qquad \widehat{\sigma}^2 = \frac{1}{K-1}\sum_{i=1}^{K}(x_i - \bar{x})^2$$

Two stochastic variables, (X_1, X_2)

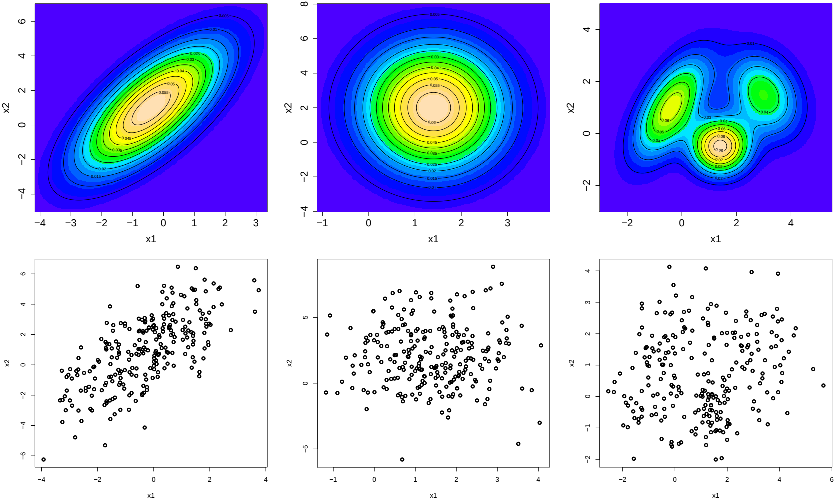
- ⋆ Concepts that are (essentially) as with one stochastic variable
  - (joint) distribution: $p(x_1, x_2)$
  - mean and variance (standard deviation)

- ⋆ New aspects
  - dependence/correlation
  - marginalisation
  - the effect of conditioning
  - how to specify $p(x_1, x_2)$

Joint distribution $p(x_1, x_2)$

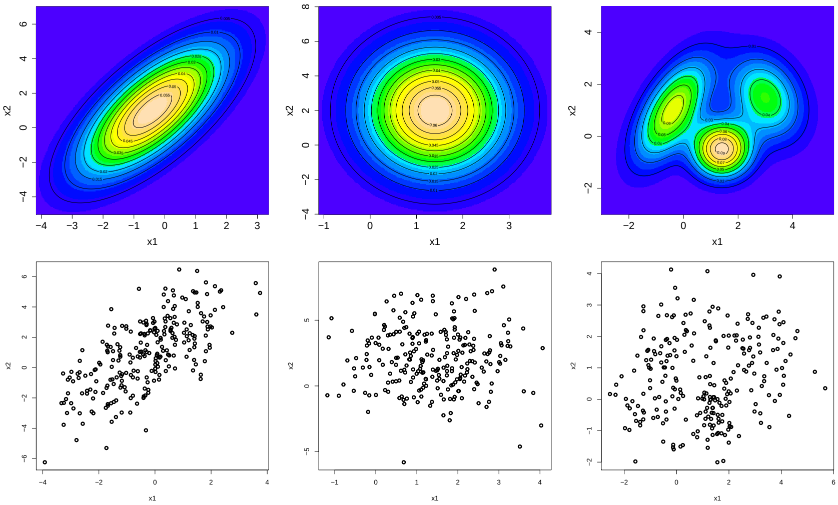


Joint distribution $p(x_1, x_2)$



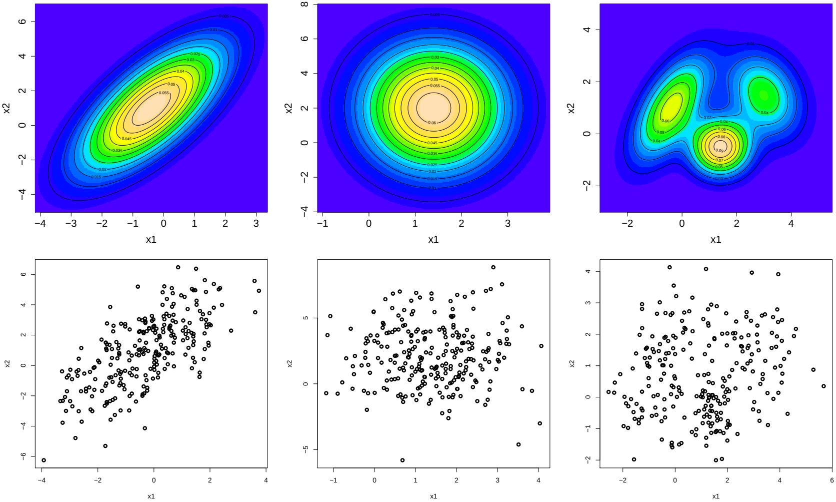
Joint distribution $p(x_1, x_2)$

- ⋆ Correlation: $\rho = \frac{\mathrm{Cov}[x_1, x_2]}{\sqrt{\mathrm{Var}[x_1] \cdot \mathrm{Var}[x_2]}} = \frac{\mathrm{E}[(x_1 - \mu_1)(x_2 - \mu_2)]}{\sqrt{\mathrm{Var}[x_1] \cdot \mathrm{Var}[x_2]}}$



Joint distribution $p(x_1, x_2)$

⋆ Correlation: $\rho = \dfrac{\mathrm{Cov}[x_1, x_2]}{\sqrt{\mathrm{Var}[x_1] \cdot \mathrm{Var}[x_2]}} = \dfrac{\mathrm{E}[(x_1 - \mu_1)(x_2 - \mu_2)]}{\sqrt{\mathrm{Var}[x_1] \cdot \mathrm{Var}[x_2]}}$



$\rho = 0.7$ $\rho = 0$ $\rho = 0.17$

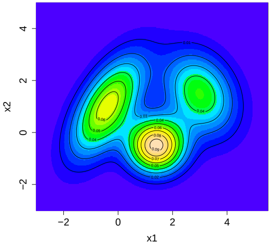
Marginalisation

- ⋆ Assume a formula for $p(x_1, x_2)$ is available
- ⋆ Marginal distribution for x_1 and for x_2

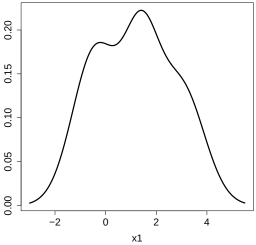
$$p(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2) dx_2$$

$$p(x_2) = \int_{-\infty}^{\infty} p(x_1, x_2) dx_1$$

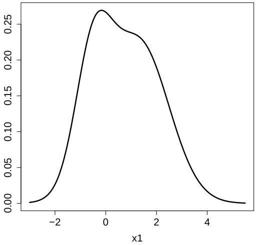
$p(x_1, x_2)$



$p(x_1)$



$p(x_2)$



- ⋆ Note: With Monte Carlo simulation marginalisation is immediate

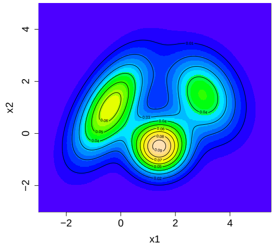
Conditioning

- ⋆ Assume a formula for $p(x_1, x_2)$
- ⋆ Conditional distributions

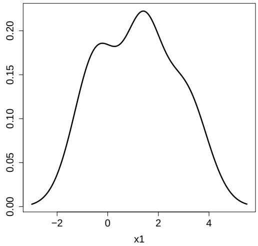
$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

$p(x_1, x_2)$



$p(x_1)$



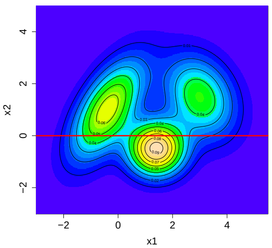
Conditioning

- ⋆ Assume a formula for $p(x_1, x_2)$
- ⋆ Conditional distributions

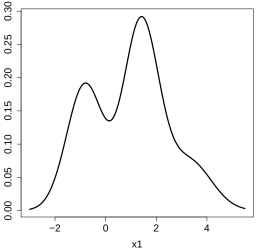
$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

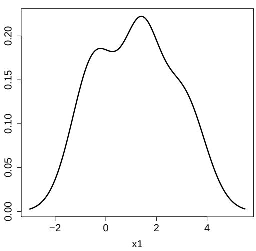
$p(x_1, x_2)$



$p(x_1|x_2 = 0)$



$p(x_1)$



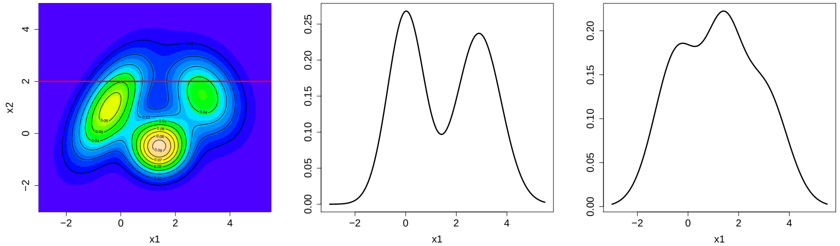
Conditioning

- ⋆ Assume a formula for $p(x_1, x_2)$
- ⋆ Conditional distributions

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

$p(x_1, x_2)$ $\quad$ $p(x_1|x_2 = 2)$ $\quad$ $p(x_1)$



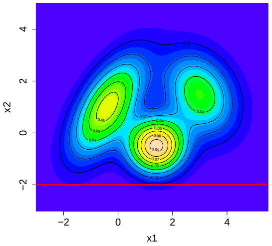
Conditioning

- ⋆ Assume a formula for $p(x_1, x_2)$
- ⋆ Conditional distributions

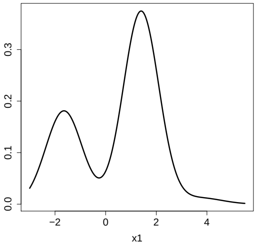
$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

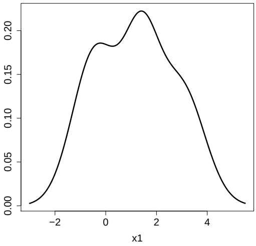
$p(x_1, x_2)$



$p(x_1|x_2 = -2)$



$p(x_1)$



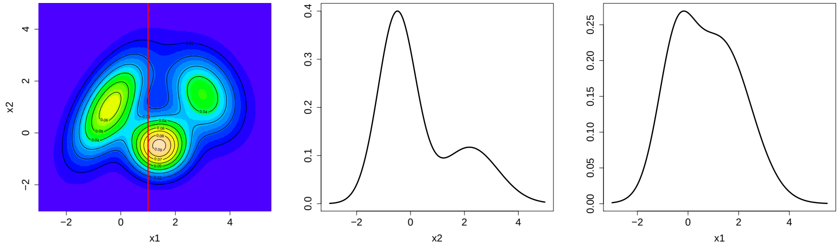
Conditioning

- ⋆ Assume a formula for $p(x_1, x_2)$
- ⋆ Conditional distributions

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

$p(x_1, x_2)$ $\quad$ $p(x_2|x_1 = 1)$ $\quad$ $p(x_2)$



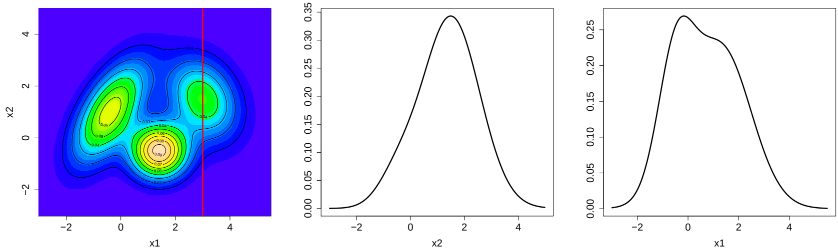
Conditioning

- ⋆ Assume a formula for $p(x_1, x_2)$
- ⋆ Conditional distributions

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

$p(x_1, x_2)$ $\quad$ $p(x_2|x_1 = 3)$ $\quad$ $p(x_2)$

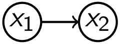


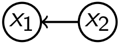
How to specify a model for two stochastic variables?

- ⋆ Specify a formula for $p(x_1, x_2)$

- ⋆ Using conditional probability

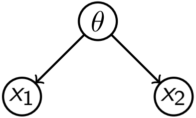
$p(x_1, x_2) = p(x_1)p(x_2|x_1)$ OR $p(x_1, x_2) = p(x_2)p(x_1|x_2)$





- ⋆ Hierarchical model

$$x_1, x_2|\theta \sim p(x|\theta) \text{ independently, and } \theta \sim p(\theta)$$



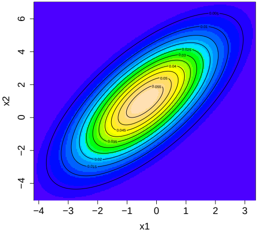
Specify a formula for $p(x_1, x_2)$

- Multivariate normal (multivariate Gaussian)

$$p(x_1, x_2) = \frac{1}{2\pi} \frac{1}{\sqrt{|\Sigma|}} \exp\left\{ -\frac{1}{2} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \mu \right)^T \Sigma^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \mu \right) \right\}$$

where

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

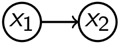


- Marginal distributions $p(x_1)$ and $p(x_2)$ are also normal

- Conditional distributions $p(x_1|x_2)$ and $p(x_2|x_1)$ are also normal

Using conditional probability to specify $p(x_1, x_2)$

- ⋆ Probability law: $p(x_1, x_2) = p(x_1)p(x_2|x_1)$

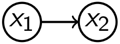


- ⋆ Simplistic example:
  - $x_1 \in \{0, 1\}$: source rock has produced hydrocarbons
  - $x_2 \in \{0, 1\}$: hydrocarbons present in prospect
  - assume probabilities: $p(x_1 = 1) = 0.3$

$$p(x_2 = 1|x_1 = 0) = 0,\ p(x_2 = 1|x_1 = 1) = 0.6$$

Using conditional probability to specify $p(x_1, x_2)$

- Probability law: $p(x_1, x_2) = p(x_1)p(x_2|x_1)$



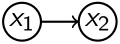
- Simplistic example:
  - $x_1 \in \{0, 1\}$: source rock has produced hydrocarbons
  - $x_2 \in \{0, 1\}$: hydrocarbons present in prospect
  - assume probabilities: $p(x_1 = 1) = 0.3$, $p(x_1 = 0) = 0.7$

$$p(x_2 = 1|x_1 = 0) = 0,\ p(x_2 = 1|x_1 = 1) = 0.6$$

$$p(x_2 = 0|x_1 = 0) = 1,\ p(x_2 = 0|x_1 = 1) = 0.4$$

Using conditional probability to specify $p(x_1, x_2)$

- Probability law: $p(x_1, x_2) = p(x_1)p(x_2|x_1)$



- Simplistic example:
  - $x_1 \in \{0, 1\}$: source rock has produced hydrocarbons
  - $x_2 \in \{0, 1\}$: hydrocarbons present in prospect
  - assume probabilities: $p(x_1 = 1) = 0.3$, $p(x_1 = 0) = 0.7$

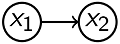
$$p(x_2 = 1|x_1 = 0) = 0,\ p(x_2 = 1|x_1 = 1) = 0.6$$
$$p(x_2 = 0|x_1 = 0) = 1,\ p(x_2 = 0|x_1 = 1) = 0.4$$

  - the joint distribution becomes

$x_1 \setminus x_2$	0	1
0	$0.7 \cdot 1 = 0.7$	$0.7 \cdot 0 = 0$
1	$0.3 \cdot 0.4 = 0.12$	$0.3 \cdot 0.6 = 0.18$

Using conditional probability to specify $p(x_1, x_2)$

- Probability law: $p(x_1, x_2) = p(x_1)p(x_2|x_1)$



- Simplistic example:
  - $x_1 \in \{0, 1\}$: source rock has produced hydrocarbons
  - $x_2 \in \{0, 1\}$: hydrocarbons present in prospect
  - assume probabilities: $p(x_1 = 1) = 0.3$, $p(x_1 = 0) = 0.7$

$$p(x_2 = 1|x_1 = 0) = 0, \; p(x_2 = 1|x_1 = 1) = 0.6$$

$$p(x_2 = 0|x_1 = 0) = 1, \; p(x_2 = 0|x_1 = 1) = 0.4$$

  - the joint distribution becomes

$x_1 \setminus x_2$	0	1
0	$0.7 \cdot 1 = 0.7$	$0.7 \cdot 0 = 0$
1	$0.3 \cdot 0.4 = 0.12$	$0.3 \cdot 0.6 = 0.18$

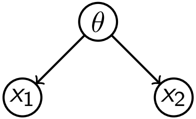
  - having observed x_2:

$$p(x_1 = 1|x_2 = 0) = \frac{p(x_1 = 1, x_2 = 0)}{p(x_2 = 0)} = \frac{0.12}{0.7 + 0.12} = 0.146$$

$$p(x_1 = 1|x_2 = 1) = \frac{p(x_1 = 1, x_2 = 1)}{p(x_2 = 1)} = \frac{0.18}{0 + 0.18} = 1$$

Hierarchical model for specifying $p(x_1, x_2)$

- Assume: $x_1, x_2|\theta \sim p(x|\theta)$ independently, $\theta \sim p(\theta)$

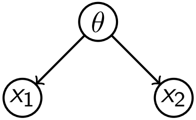


- Simplistic example:
  - $\theta \in \{0, 1\}$: source rock has produced hydrocarbons
  - $x_1 \in \{0, 1\}$: hydrocarbons present in prospect 1
  - $x_2 \in \{0, 1\}$: hydrocarbons present in prospect 2
  - assume probabilities: $p(\theta = 1) = 0.3$

$$p(x_1 = 1|\theta = 0) = 0, \; p(x_1 = 1|\theta = 1) = 0.6$$
$$p(x_2 = 1|\theta = 0) = 0, \; p(x_2 = 1|\theta = 1) = 0.6$$

Hierarchical model for specifying $p(x_1, x_2)$

⋆ Assume: $x_1, x_2|\theta \sim p(x|\theta)$ independently, $\theta \sim p(\theta)$



⋆ Simplistic example:

- $\theta \in \{0, 1\}$: source rock has produced hydrocarbons
- $x_1 \in \{0, 1\}$: hydrocarbons present in prospect 1
- $x_2 \in \{0, 1\}$: hydrocarbons present in prospect 2
- assume probabilities: $p(\theta = 1) = 0.3$

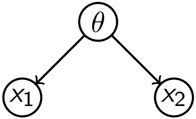
$$p(x_1 = 1|\theta = 0) = 0, \; p(x_1 = 1|\theta = 1) = 0.6$$
$$p(x_2 = 1|\theta = 0) = 0, \; p(x_2 = 1|\theta = 1) = 0.6$$

- joint distribution:

$$p(\theta = 0, x_1 = 0, x_2 = 0) = 0.7 \cdot 1 \cdot 1 = 0.7$$
$$\vdots$$
$$p(\theta = 1, x_1 = 1, x_2 = 0) = 0.3 \cdot 0.6 \cdot 0.4 = 0.072$$
$$p(\theta = 1, x_1 = 1, x_2 = 1) = 0.3 \cdot 0.6 \cdot 0.6 = 0.108$$

Hierarchical model — simplistic example

- Recall assumed model:



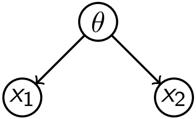
$$p(\theta = 1) = 0.3$$
$$p(x_1 = 1|\theta = 0) = 0, \quad p(x_1 = 1|\theta = 1) = 0.6$$
$$p(x_2 = 1|\theta = 0) = 0, \quad p(x_2 = 1|\theta = 1) = 0.6$$

- From the joint distribution we can get (for example)

$$p(x_1 = 1) = \sum_{\theta=0}^{1} \sum_{x_2=0}^{1} p(\theta, x_1 = 1, x_2) = 0.18$$

Hierarchical model — simplistic example

- Recall assumed model:



$$p(\theta = 1) = 0.3$$
$$p(x_1 = 1|\theta = 0) = 0, \quad p(x_1 = 1|\theta = 1) = 0.6$$
$$p(x_2 = 1|\theta = 0) = 0, \quad p(x_2 = 1|\theta = 1) = 0.6$$

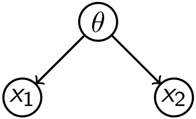
- From the joint distribution we can get (for example)

$$p(x_1 = 1) = \sum_{\theta=0}^{1} \sum_{x_2=0}^{1} p(\theta, x_1 = 1, x_2) = 0.18$$

$$p(x_1 = 1|x_2 = 0) = \frac{p(x_1 = 1, x_2 = 0)}{p(x_2 = 0)}$$
$$= \frac{\sum_{\theta=0}^{1} p(\theta, x_1 = 1, x_2 = 0)}{\sum_{\theta=0}^{1} \sum_{x_1=0}^{1} p(\theta, x_1, x_2 = 0)} = 0.0878$$

Hierarchical model — simplistic example

- Recall assumed model:



$$p(\theta = 1) = 0.3$$
$$p(x_1 = 1|\theta = 0) = 0, \quad p(x_1 = 1|\theta = 1) = 0.6$$
$$p(x_2 = 1|\theta = 0) = 0, \quad p(x_2 = 1|\theta = 1) = 0.6$$

- From the joint distribution we can get (for example)

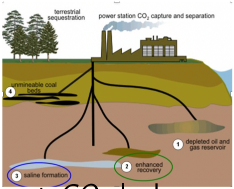
$$p(x_1 = 1) = \sum_{\theta=0}^{1} \sum_{x_2=0}^{1} p(\theta, x_1 = 1, x_2) = 0.18$$

$$p(x_1 = 1|x_2 = 0) = \frac{p(x_1 = 1, x_2 = 0)}{p(x_2 = 0)}$$
$$= \frac{\sum_{\theta=0}^{1} p(\theta, x_1 = 1, x_2 = 0)}{\sum_{\theta=0}^{1} \sum_{x_1=0}^{1} p(\theta, x_1, x_2 = 0)} = 0.0878$$

$$p(x_1 = 1|x_2 = 1) = \ldots = 0.6$$

CO_2 leakage example and decision making

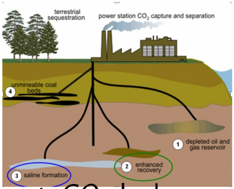
- ⋆ Situation: Company can
  - *i*) proceed with CO_2 injection
    - cost of 30 monetary units
    - if CO_2 leakage, fine of 60 monetary units
  - *ii*) suspend sequestration
    - tax of 80 monetary units



- ⋆ Can use seismics to get (noisy) information about CO_2 leakage
- ⋆ What decision should the company do?
  - do seismics or not do seismics
  - inject CO_2 or suspend sequestration
- ⋆ Model assumptions
  - $x \in \{0, 1\}$: CO_2 leakage, $p(x = 1) = 0.3$
  - $y \in \{0, 1\}$: seismic information, $p(y = 1|x = 1) = 0.9$, $p(y = 0|x = 0) = 0.9$

CO_2 leakage example and decision making

- ⋆ Situation: Company can
  - *i*) proceed with CO_2 injection
    - cost of 30 monetary units
    - if CO_2 leakage, fine of 60 monetary units
  - *ii*) suspend sequestration
    - tax of 80 monetary units



- ⋆ Can use seismics to get (noisy) information about CO_2 leakage
- ⋆ What decision should the company do?
  - do seismics or not do seismics
  - inject CO_2 or suspend sequestration
- ⋆ Model assumptions
  - $x \in \{0, 1\}$: CO_2 leakage, $p(x = 1) = 0.3$
  - $y \in \{0, 1\}$: seismic information, $p(y = 1|x = 1) = 0.9$, $p(y = 0|x = 0) = 0.9$
- ⋆ Expected cost if injection

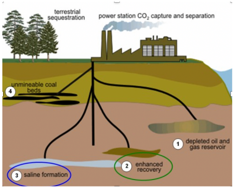
$$\text{E[cost injection]} = 30 \cdot p(x = 0) + (30 + 60) \cdot p(x = 1) = 48$$

CO_2 leakage example and decision making

- ⋆ Situation: Company can

  i) proceed with CO_2 injection
  - cost of 30 monetary units
  - if CO_2 leakage, fine of 60 monetary units

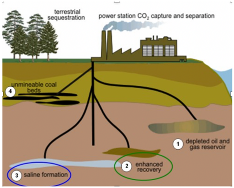
  ii) suspend sequestration
  - tax of 80 monetary units



- ⋆ Model assumptions
  - $x \in \{0,1\}$: CO_2 leakage, $p(x=1) = 0.3$
  - $y \in \{0,1\}$: seismic information, $p(y=1|x=1) = 0.9$, $p(y=0|x=0) = 0.9$
- ⋆ With seismic information

CO_2 leakage example and decision making

- ⋆ Situation: Company can
  - *i*) proceed with CO_2 injection
    - – cost of 30 monetary units
    - – if CO_2 leakage, fine of 60 monetary units
  - *ii*) suspend sequestration
    - – tax of 80 monetary units



- ⋆ Model assumptions
  - – $x \in \{0, 1\}$: CO_2 leakage, $p(x = 1) = 0.3$
  - – $y \in \{0, 1\}$: seismic information, $p(y = 1|x = 1) = 0.9$, $p(y = 0|x = 0) = 0.9$
- ⋆ With seismic information

$$p(x = 1|y = 0) = \frac{p(x = 1)p(y = 0|x = 1)}{p(y = 0)} = 0.045$$

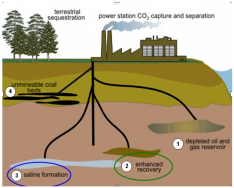
$$p(x = 1|y = 1) = \frac{p(x = 1)p(y = 1|x = 1)}{p(y = 1)} = 0.794$$

CO_2 leakage example and decision making

⋆ Situation: Company can

i) proceed with CO_2 injection
  - cost of 30 monetary units
  - if CO_2 leakage, fine of 60 monetary units

ii) suspend sequestration
  - tax of 80 monetary units



⋆ Model assumptions
  - $x \in \{0, 1\}$: CO_2 leakage, $p(x = 1) = 0.3$
  - $y \in \{0, 1\}$: seismic information, $p(y = 1|x = 1) = 0.9$, $p(y = 0|x = 0) = 0.9$

⋆ With seismic information

$$p(x = 1|y = 0) = \frac{p(x = 1)p(y = 0|x = 1)}{p(y = 0)} = 0.045$$

$$p(x = 1|y = 1) = \frac{p(x = 1)p(y = 1|x = 1)}{p(y = 1)} = 0.794$$

⋆ Expected costs

$$E[\text{cost injection}|y = 0] = 30 \cdot (1 - 0.045) + (30 + 60) \cdot 0.045 = 32.7$$

$$E[\text{cost injection}|y = 1] = 30 \cdot (1 - 0.794) + (30 + 60) \cdot 0.794 = 77.6$$

Many (n) stochastic variables

- ⋆ Concepts that are (essentially) as with two stochastic variables
  - (joint) distribution
  - mean and variance (standard deviation)
  - dependence/correlation
  - marginalisation
  - the effect of conditioning
  - how to specify the distribution

- ⋆ New aspects
  - conditional independence
  - how to "look at" the distribution
  - what quantities are of interest
  - how to generate Monte Carlo realizations/samples