Value of Information Analysis in Spatial Models

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Plan for course

Time	**Topic**
Monday	Introduction and motivating examples
	Elementary decision analysis and the value of information
Tuesday	Multivariate statistical modeling, dependence, graphs
	Value of information analysis for dependent models
Wednesday	Spatial statistics, spatial design of experiments
	Value of information analysis in spatial decision situations
Thursday	Examples of value of information analysis in Earth sciences
	Computational aspects
Friday	Sequential decisions and sequential information gathering
	Examples from mining and oceanography

Every day: Small exercise half-way, and computer project at the end.

Formula for VOI

$$PV = \max_{\boldsymbol{a}\in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x},\boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a}\in \boldsymbol{A}} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x},\boldsymbol{a})\, p\left(\boldsymbol{x} \right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{y} \right) = \int \max_{\boldsymbol{a}\in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x},\boldsymbol{a} \right) \mid \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y} \right) d\boldsymbol{y}$$

$$VOI\left(\boldsymbol{y} \right) = PoV\left(\boldsymbol{y} \right) - PV.$$

Formula for VOI

Spatial alternatives.

Spatial uncertainties.

$$PV = \max_{\boldsymbol{a} \in A} \left\{ E\left(v(\boldsymbol{x},\boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a} \in A} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x},\boldsymbol{a}) p\left(\boldsymbol{x}\right) d\boldsymbol{x} \right\}$$

Spatial value function.

$$PoV\left(\boldsymbol{y}\right) = \int \max_{\boldsymbol{a} \in A} \left\{ E\left(v\left(\boldsymbol{x},\boldsymbol{a}\right) \mid \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

Spatial data.

$$VOI\left(\boldsymbol{y}\right) = PoV\left(\boldsymbol{y}\right) - PV.$$

Information gathering

	Perfect	Imperfect
Total	Exact observations are gathered for all locations. This is rare, occurring when there is extensive coverage and highly accurate data gathering. $$\boldsymbol{y} = \boldsymbol{x}$$	Noisy observations are gathered for all locations. This is common in situations with remote sensors with extensive coverage, e.g. seismic, radar, satellite data. $$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\varepsilon}$$
Partial	Exact observations are gathered at some locations. This might occur, for instance, when there is careful analysis of rock samples along boreholes in a reservoir or a mine. $$\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}}, \quad \mathbb{K} \text{ subset}$$	Noisy observations are gathered at some locations. Examples include hand-held (noisy) meters to observe grades in mine boreholes, electromagnetic testing along a line, biological surveys of species, etc. $$\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}} + \boldsymbol{\varepsilon}_{\mathbb{K}}, \quad \mathbb{K} \text{ subset}$$

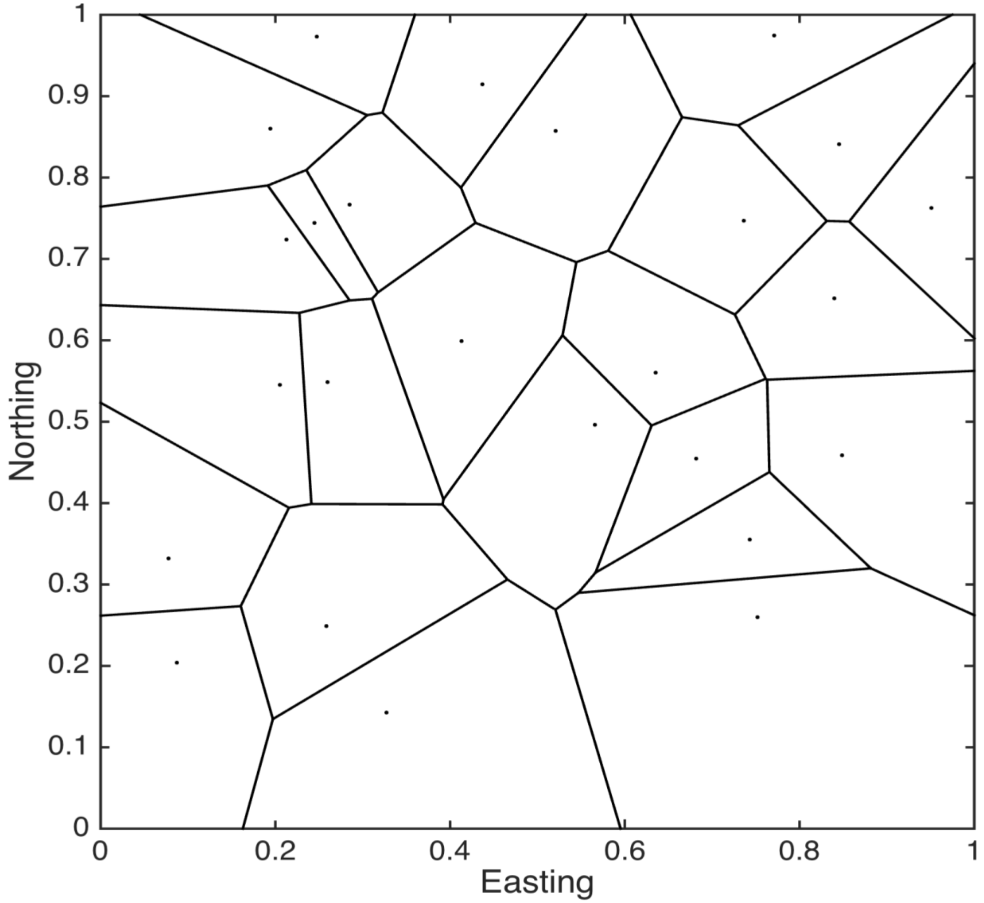
Decision situations and values

	Assumption: Decision Flexibility	**Assumption: Value Function**
Low decision flexibility; Decoupled value	Alternatives are easily enumerated $a \in A$	Total value is a sum of value at every unit $v(\mathbf{x}, a) = \sum_j v(x_j, a)$
High decision flexibility; Decoupled value	None $\boldsymbol{a} \in \boldsymbol{A}$	Total value is a sum of value at every unit $v(\mathbf{x}, \boldsymbol{a}) = \sum_j v(x_j, a_j)$
Low decision flexibility; Coupled value	Alternatives are easily enumerated $a \in A$	None $v(\mathbf{x}, a)$
High decision flexibility; Coupled value	None $\boldsymbol{a} \in \boldsymbol{A}$	None $v(\mathbf{x}, \boldsymbol{a})$

Low versus high decision flexibility

High flexibility:
Farmer can select individual forest units.

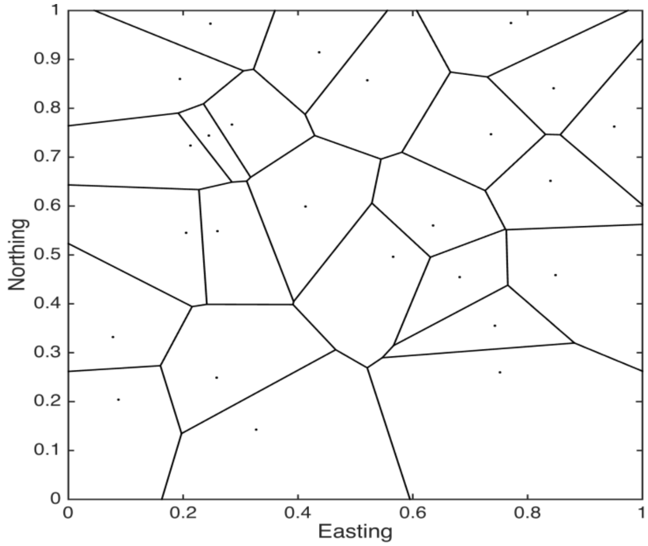
Low flexibility:
Farmer must select all forest units, or none.



Decoupled versus coupled value

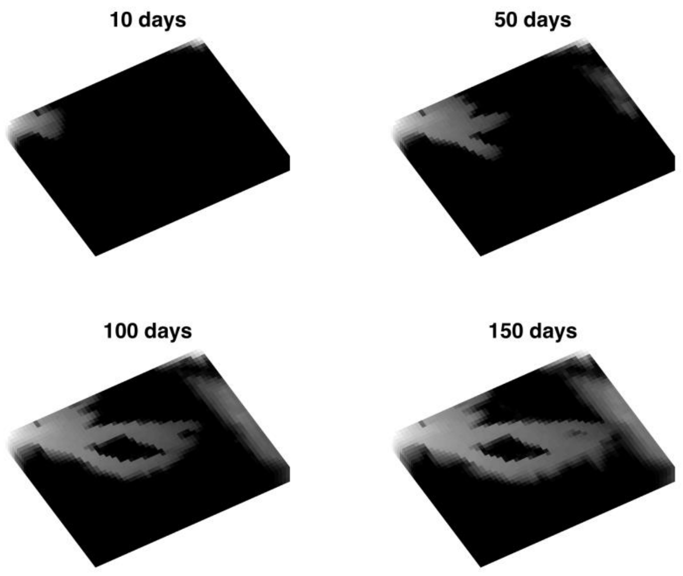
Farmer must decide whether to harvest at forest units, or not.

Value decouples to sum over units.



Petroleum company must decide how to produce a reservoir.

Value involves complex coupling of drilling strategies, and reservoir properties.



Computation - Formula for VOI

$$PV = \max_{\boldsymbol{a}\in \boldsymbol{A}}\left\{E\left(v(\boldsymbol{x},\boldsymbol{a})\right)\right\} = \max_{\boldsymbol{a}\in \boldsymbol{A}}\left\{\int_{\boldsymbol{x}} v(\boldsymbol{x},\boldsymbol{a})\, p\left(\boldsymbol{x}\right) d\boldsymbol{x}\right\}$$

$$PoV\left(\boldsymbol{y}\right) = \int \max_{\boldsymbol{a}\in \boldsymbol{A}}\left\{E\left(v\left(\boldsymbol{x},\boldsymbol{a}\right) \mid \boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

Main challenge.

Computations :

- Easier with low decision flexibility (less alternatives).
- Easier if value decouples (sums or integrals split).
- Easier for perfect, total, information (upper bound on VOI).
- Sometimes analytical solutions, otherwise approximations and Monte Carlo.

Techniques – Computing the VOI

$$PV = \max_{\boldsymbol{a} \in A} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a} \in A} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{a}) p(\boldsymbol{x}) d\boldsymbol{x} \right\}$$

$$PoV(\boldsymbol{y}) = \int \max_{\boldsymbol{a} \in A} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \mid \boldsymbol{y} \right) \right\} p(\boldsymbol{y}) d\boldsymbol{y}$$

Inner integral.

Outer integral.

Techniques :

- Fully analytically tractable for two-action, Gaussian, linear models.
- Analytical or partly analytical for Markovian models, graphs.
- (Monte Carlo sample over data, analytical for inner expectation.)
- Various approximations and Monte Carlo usually applicable.
- Should avoid double Monte Carlo (inner and outer). Too time consuming.

Fully analytical

$$PV = \max\{0, \mu_w\}$$

Inner integral analytical.
Linear combination of data.

$$PoV(\boldsymbol{y}) = \int \max\{0, \boldsymbol{\alpha}'\boldsymbol{y}\}\, p(\boldsymbol{y})\, d\boldsymbol{y}$$

Gaussian.

$$= \int \max\{0, w\}\, p(w)\, dv = \mu_w \Phi\left(\frac{\mu_w}{r_w}\right) + r_w \phi\left(\frac{\mu_w}{r_w}\right)$$

Partly analytical, Monte Carlo for rest

$$PV = \max\{0, \mu_w\}$$

Inner integral solved.

$$PoV(\boldsymbol{y}) = \int \max\{0, f(\boldsymbol{y})\}\, p(\boldsymbol{y})\, d\boldsymbol{y}$$

Use sampling.

$$= \frac{1}{B}\sum_{b=1}^{B} \max\{0, f(\boldsymbol{y}^b)\}$$

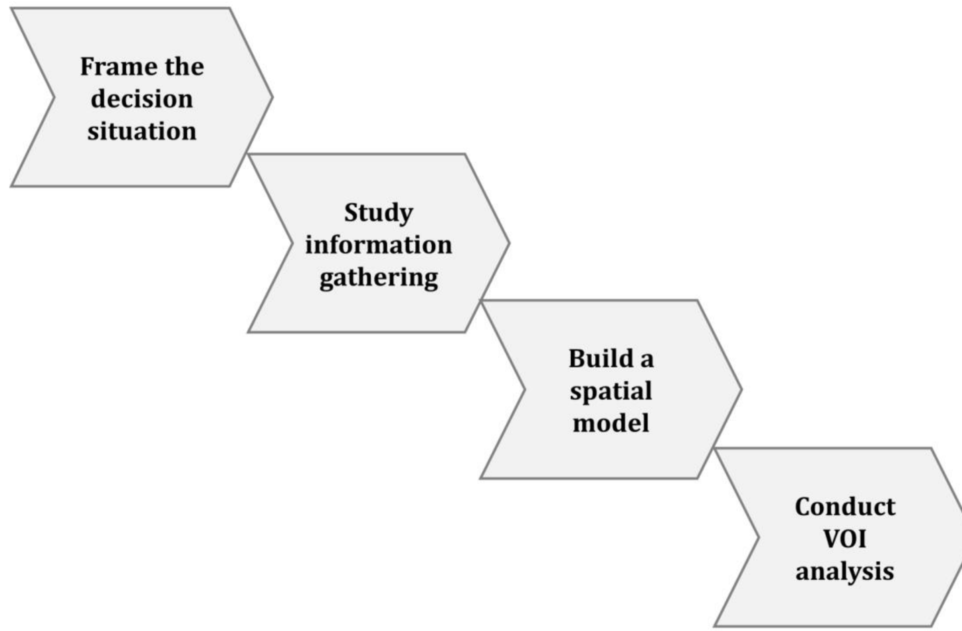
$$\boldsymbol{y}^b \sim p(\boldsymbol{y}), \qquad b = 1, \ldots, B.$$

Examples of other approximations

- Rock hazard example in mining : Laplace approximations.
- Petroleum production example : Monte Carlo simulation and regression.

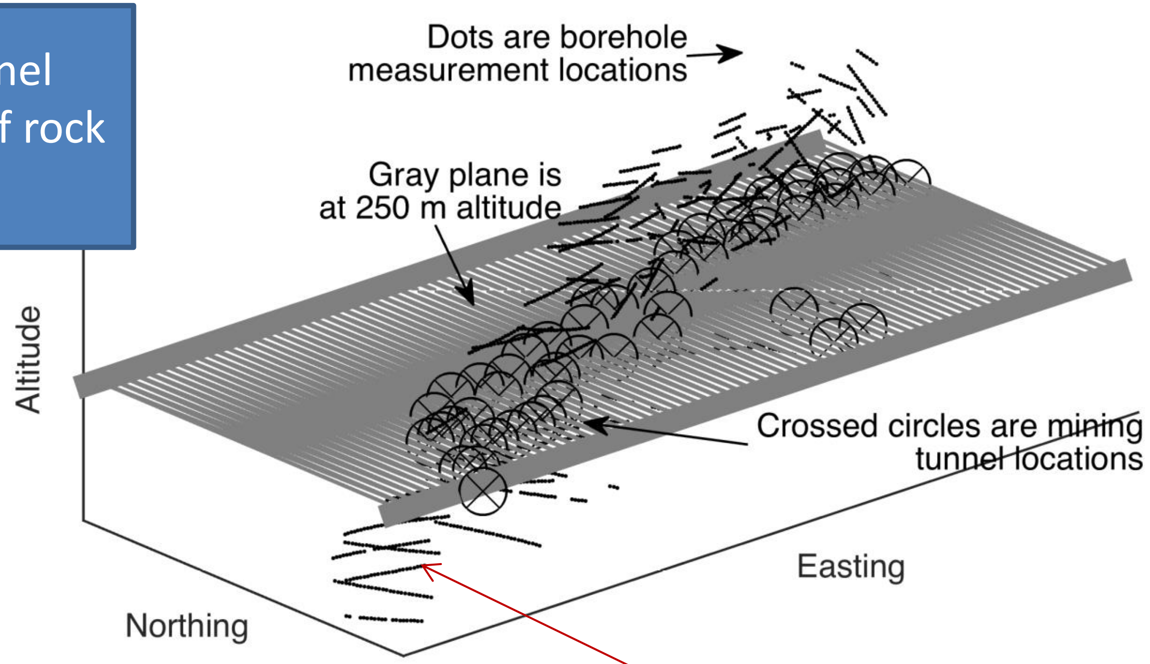
We will rock you – rock hazard example

- Decisions about stabilizing rock mass.
- Borehole information. How much and where?
- Model is a represented by Gaussian process, with Poisson count measurements.
- VOI analysis done by a Gaussian approximations and Laplace approximations.



We will rock you – Spatial GLM

Add support at tunnel locations where risk of rock fall is too high.



What is the value of additional borehole information about spatial joint counts?

We will rock you – Model

Risk of rock fall is tied to the number of joints in rocks.

$$\boldsymbol{x} = \left(x_1, \ldots, x_n \right)$$

$$p\left(\boldsymbol{x} \right) = N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma} \right)$$

Spatial joint intensity.

$$p\left(y\left(\mathbf{s}_i \right) \mid x\left(\mathbf{s}_i \right) \right) = \text{Poisson}\left(V_i \exp\left(x\left(\mathbf{s}_i \right) \right) \right)$$

$$\boldsymbol{y} = \left(y_1, \ldots, y_m \right)$$

Data are collected in boreholes.
Design of boreholes is important.

Values

$$PV = \sum_{j=1}^{52} \max\left\{-\text{Cost1}, -\text{Cost2} \cdot E\left(\exp\left(x\left(\mathbf{s}_j\right)\right)\right)\right\},$$

Costs of adding supports, and costs if support is not added and rock collapse.

$$PoV\left(\boldsymbol{y}\right) = \sum_{\boldsymbol{y}} \sum_{j=1}^{52} \max\left\{-\text{Cost1}, -\text{Cost2} \cdot E\left(\exp\left(x\left(\mathbf{s}_j\right)\right) \mid \boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right)$$

Approximations

$$PoV(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \sum_{j=1}^{52} \max\left\{-\text{Cost1}, -\text{Cost2} \cdot E\left(\exp\left(x\left(\mathbf{s}_j\right)\right) \mid \boldsymbol{y}\right)\right\} p(\boldsymbol{y})$$

$$\hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} p(\boldsymbol{y} \mid \boldsymbol{x}) p(\boldsymbol{x})$$

$$E\left(x_j \mid \boldsymbol{y}\right) \approx \mu_j + \boldsymbol{\Sigma}_{j,*} \boldsymbol{\Sigma}^{-1} (\hat{\boldsymbol{x}} - \boldsymbol{\mu}),$$

$$Var\left(x_j \mid \boldsymbol{y}\right) \approx \sigma_j^2 - \boldsymbol{\Sigma}_{j,*} \left(\boldsymbol{\Sigma} + \hat{\boldsymbol{D}}^{-1}\right)^{-1} \boldsymbol{\Sigma}_{j,*}^t$$

Gaussian approximation, Laplace approximation, and matrix approximation.

$$(\boldsymbol{I} + \varepsilon \boldsymbol{A})^{-1} \approx \boldsymbol{I} - \varepsilon \boldsymbol{A}$$

$$Var(\hat{\boldsymbol{x}}) = \boldsymbol{R} = \boldsymbol{\Sigma} \boldsymbol{F}^t E\left(\boldsymbol{F} \boldsymbol{\Sigma} \boldsymbol{F}^t + \hat{\boldsymbol{D}}^{-1}\right)^{-1} \boldsymbol{F} \boldsymbol{\Sigma} \approx \boldsymbol{\Sigma} \boldsymbol{F}^t \left(\boldsymbol{F} \boldsymbol{\Sigma} \boldsymbol{F}^t + E\left(\hat{\boldsymbol{D}}^{-1}\right)\right)^{-1} \boldsymbol{F} \boldsymbol{\Sigma}$$

Evangelou and Eidsvik, 2017, The Value of Information for correlated GLMs, JSPI, p 30-48.

Approximations (ii)

$$\sum_{\boldsymbol{y}} \sum_{j=1}^{N} \max\left\{0, \text{Cost1} \cdot E\left(\exp\left(x_j\right) \mid \boldsymbol{y}\right) - \text{Cost2}\right\} p\left(\boldsymbol{y}\right)$$

$$\approx \sum_{j=1}^{N} \int \max\left\{0, \text{Cost1} \cdot \exp\left(\hat{x}_j + \frac{\xi_j^2}{2}\right) - \text{Cost2}\right\} p\left(\hat{x}_j\right) d\hat{x}_j$$

$$\approx \sum_{j=1}^{N} \exp\left(\frac{\xi_j^2}{2}\right) \text{Cost1} \int_{e_j}^{\infty} \exp\left(\hat{x}_j\right) p\left(\hat{x}_j\right) d\hat{x}_j - \text{Cost2} \int_{e_j}^{\infty} p\left(\hat{x}_j\right) d\hat{x}_j$$

$$= \text{Cost1} \cdot \exp\left(\mu_j + \xi_j^2 / 2 + r_j^2 / 2\right) \Phi\left(\frac{e_j - \mu_j - r_j^2}{r_j}\right) - \text{Cost2} \cdot \Phi\left(\frac{e_j - \mu_j}{r_j}\right)$$

Evangelou and Eidsvik, 2017, The Value of Information for correlated GLMs, JSPI, p 30-48.

Approximations (ii)

Exercise:

1. **Compute the log Gaussian expectation.**

$$y = \exp(x), \qquad p(x) = N\left(\mu, \sigma^2\right)$$

$$E(y) = E\left(\exp(x)\right) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

Results – data gathering options

$$VOI(\mathbf{y}) = PoV(\mathbf{y}) - PV.$$

Study the value of different borehole information, and compared with the prices.

- Test half of planned boreholes, with half of samples (every other).
- Test quarter of boreholes, with dense sampling.
- Do no testing.

Results – decision regions

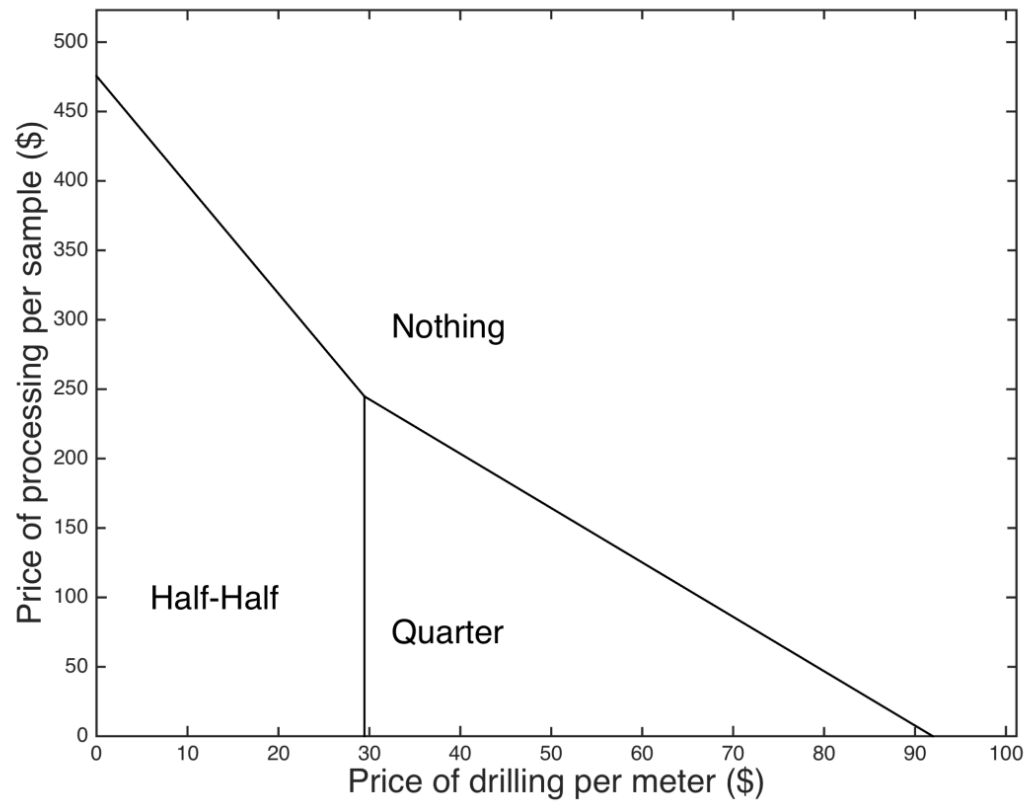
No testing.

$$\arg\max\left\{0, \mathrm{VOI}_1 - P_1, \mathrm{VOI}_2 - P_2\right\}$$

We here gather the data that is most valuable compared with its price.

This data gathering decision could also be made according to a budget. Then we collect data that has the largest VOI, as long as it is less than the budget.

Results



$$\arg\max\left\{0, \mathrm{VOI}_{\text{Half-half}} - P_{\text{Half-half}}, \mathrm{VOI}_{\text{Quarter}} - P_{\text{Quarter}}\right\}$$

P1 and P2 depend on the length of boreholes, and number of tests.
The number of tests are here the same for the two data gathering options.
Half-half drills about twice as long as the Quarter option.

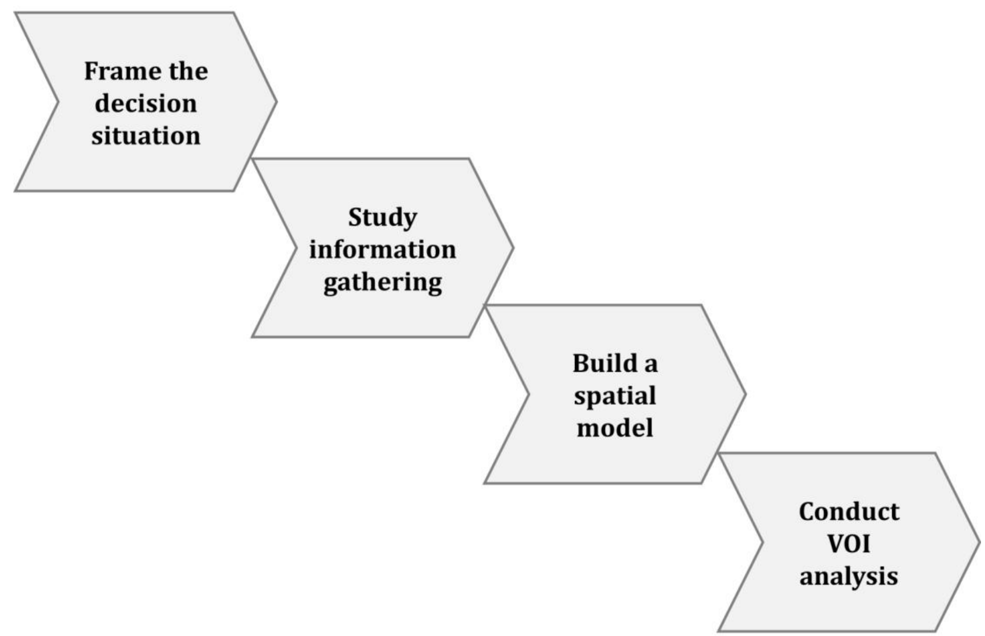
Wrap up

- Analytical expressions allows fast computations, and eases sensitivity analysis, etc.
- Approximations must be checked (time consuming MCMC, asymptotics, etc.)

- VOI of subset testing can be effective when there is correlation. Two dense data samples does not give double information.

Reservoir dogs - petroleum example

- Decisions about drilling alternatives.
- Seismic information. Which kind?
- Model is a represented by spatial process, obtained by simulations.
- VOI analysis done by a simulation-regression approach.



Fluid flow modeling example

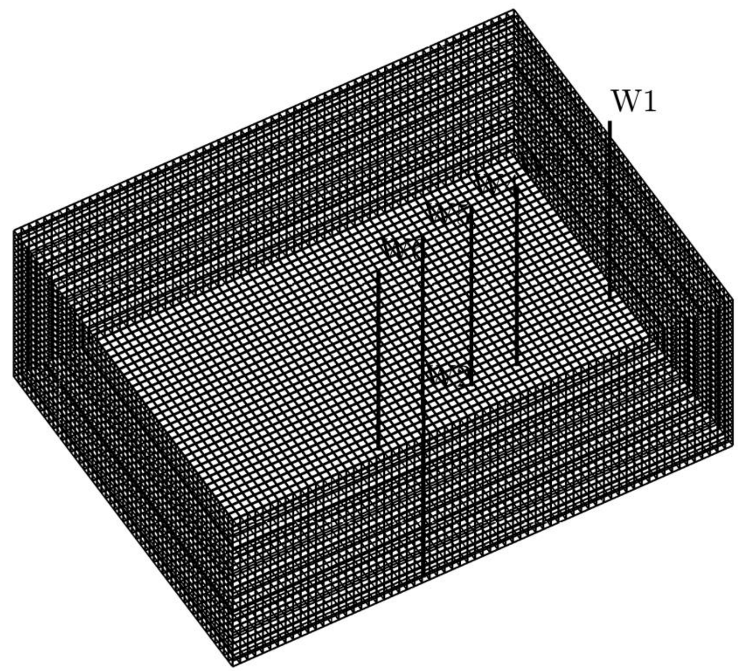
Most reservoir decisions require reservoir simulation, fluid flow is important for the value calculcation.

Develop with a drilling plan, or not develop at all:

$$a = \{0,..,9\}$$

Value computed from petroleum reservoir simulation.

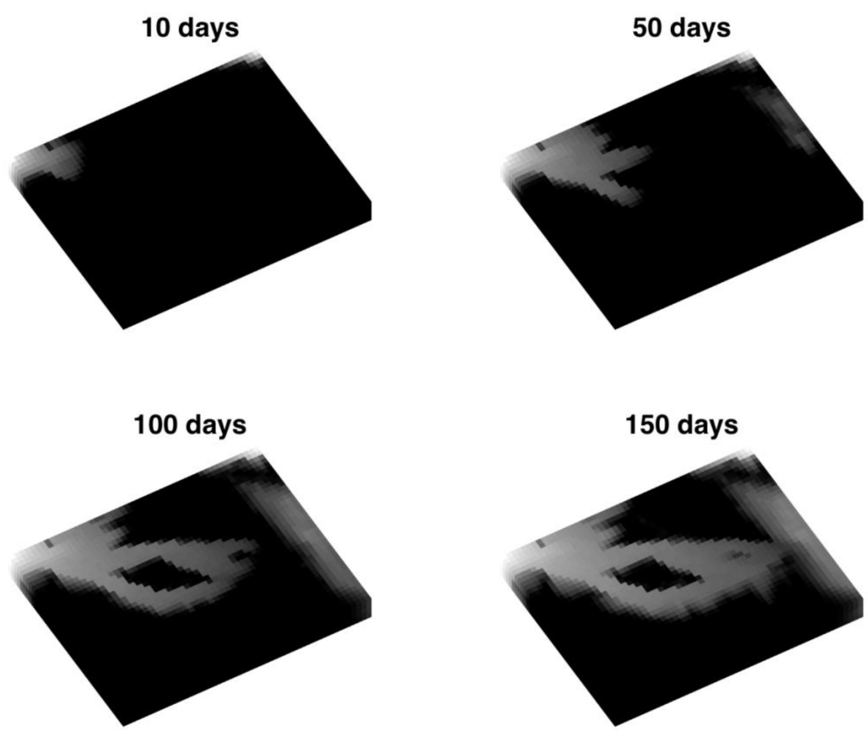
$$v(\boldsymbol{x}, a)$$



Dutta et al., 2017.

Reservoir flow in heterogeneous media

Injection of water, pushing oil out.



Flow in the reservoir depends on the composition of rocks, porosity, permeability, faults, etc.

Seismic data can help identify these important reservoir properties.

Very non-linear relations!

Computation - Formula for VOI

$$PV = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{a}) p\left(\boldsymbol{x} \right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{y} \right) = \int \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x}, \boldsymbol{a} \right) \mid \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y} \right) d\boldsymbol{y}$$

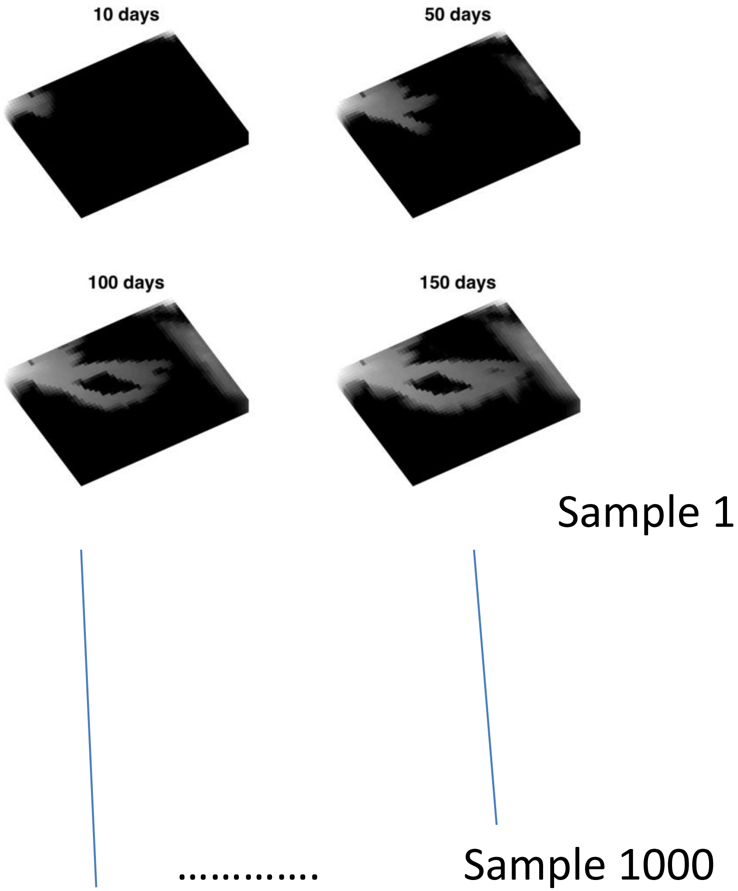
Computationally challenging:

- No decoupling in space. Joint optimization over all alternatives.
- Non-linear value function and seismic data.

Prior - Reservoir uncertainty

Prior is $p(\boldsymbol{x})$.

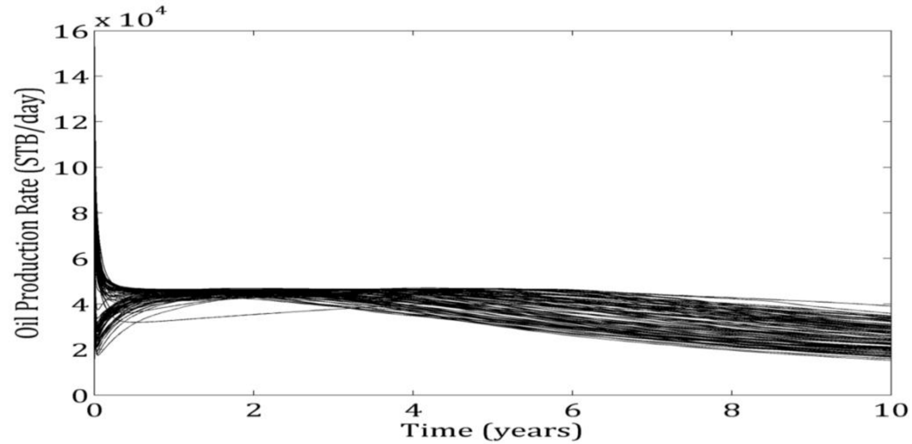
This distribution of reservoir variables is usually represented by multiple Monte Carlo realizations from the prior distribution.

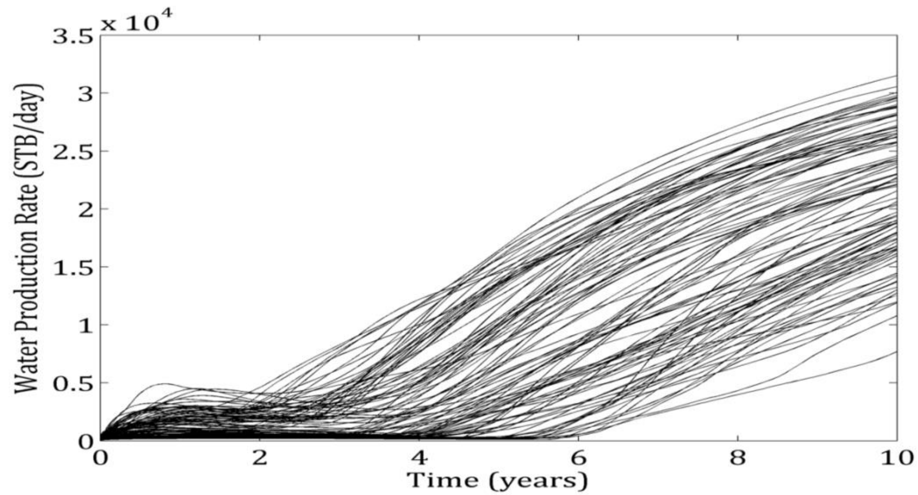


Flow simulation and value

$$v\left(\boldsymbol{x}^b, \boldsymbol{a}\right), \qquad b = 1, \ldots, 1000$$

Flow simulation gives amount of recoverable oil, for each realization, with the development cost subtracted.

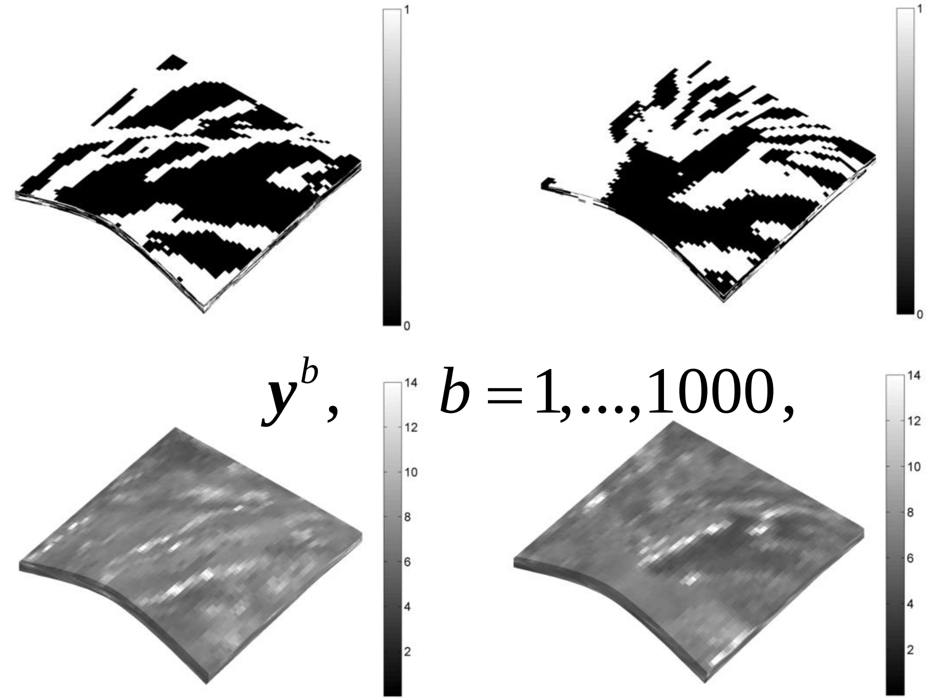




Likelihood - Seismic data

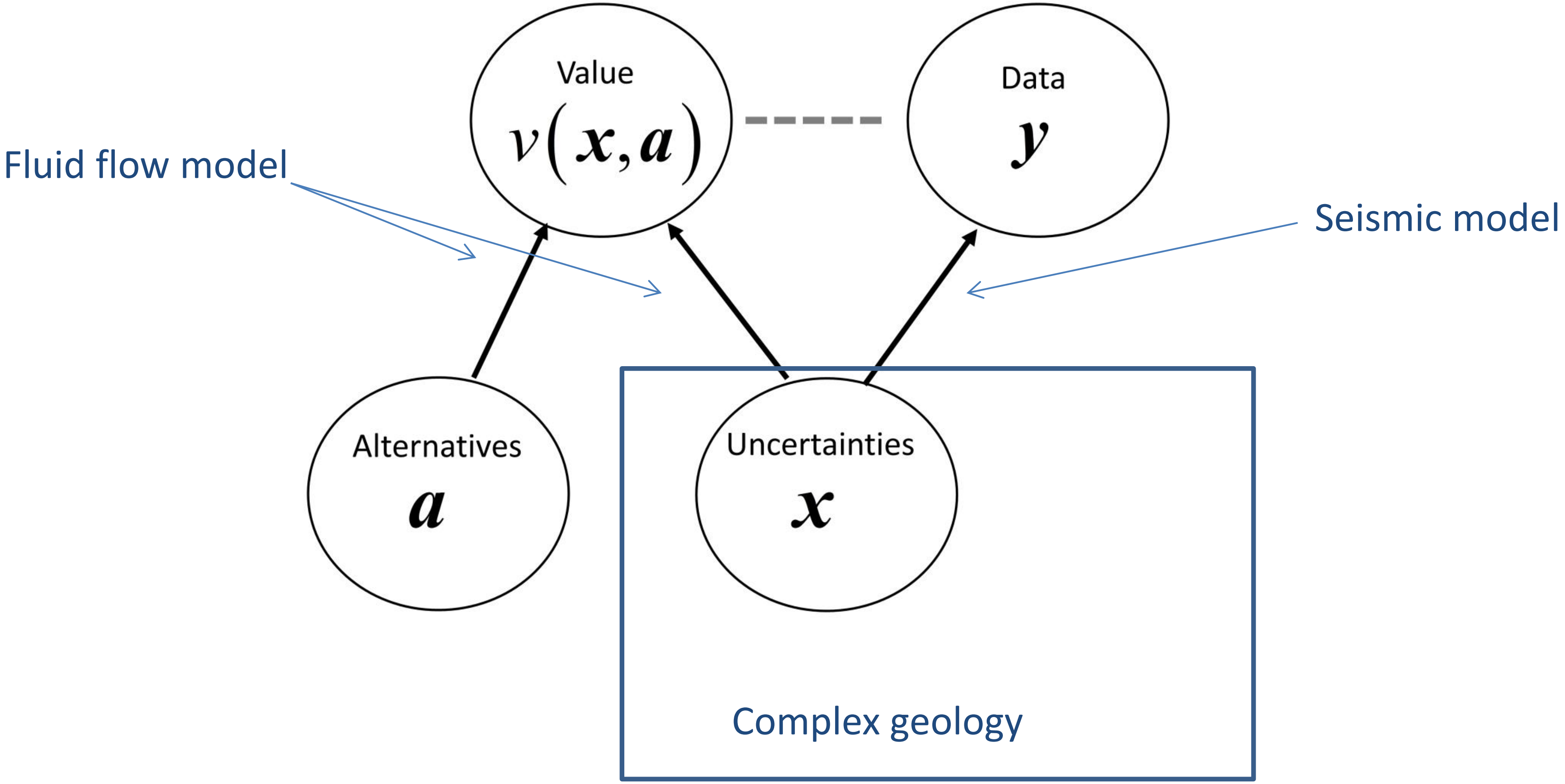
$$p(\boldsymbol{y} \mid \boldsymbol{x}) = N(\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{T})$$

The likelihood is non-linear, but we can generate synthetic seismic data from the likelihood model, given realizations of reservoir properties.



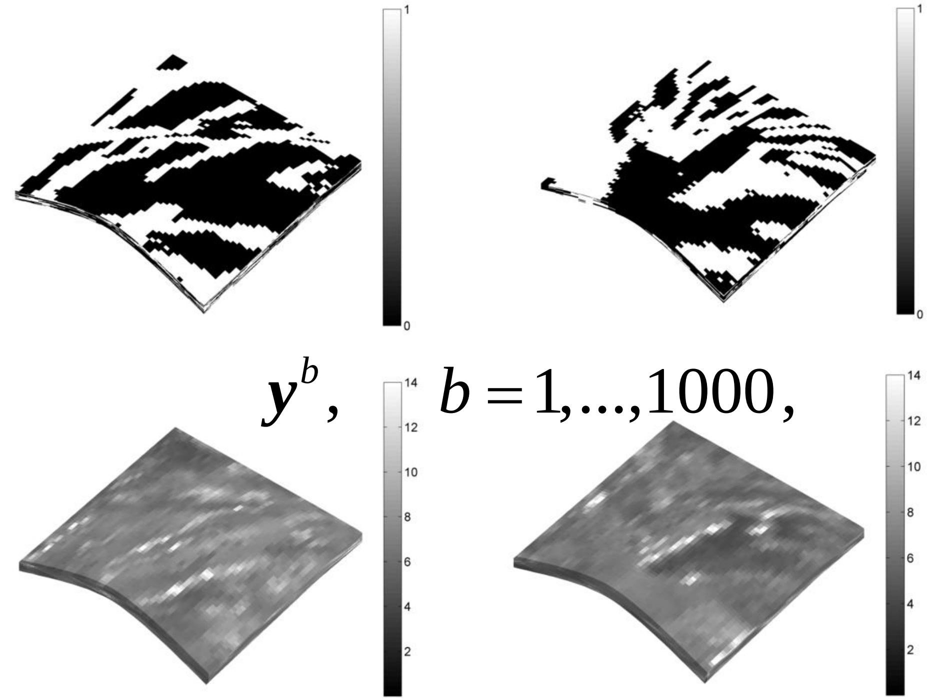
$$\boldsymbol{y}^b, \quad b = 1, \ldots, 1000,$$

Linking variables



Value and seismic data

- Random draws of geologic scenario (meandering channels or delta):
- Draw rock-type realizations
- Draw porosity realizations
- Draw permeability realizations
- **Draw value** by fluid flow simulation and economics
- **Draw seismic data** using physics



$$\boldsymbol{y}^b, \quad b = 1, \ldots, 1000,$$

We next use these samples for VOI approximation.

Computation

Outer expectation: $\boldsymbol{y}$

$$PoV(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \mid \boldsymbol{y} \right) \right\} p(\boldsymbol{y})$$

Inner expectation: $\boldsymbol{x} \mid \boldsymbol{y}$

$$VOI(\boldsymbol{y}) = PoV(\boldsymbol{y}) - PV$$

- Monte Carlo (outer) and simulation-regression for inner expectation!

Simulation-regression algorithm

Outer expectation

$$PoV(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \max_{\boldsymbol{a}\in\boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x},\boldsymbol{a}\right) \mid \boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right)$$

Inner expectation

1. Simulate uncertainties: $\boldsymbol{x}^b, \quad b = 1,\ldots,B$
2. Simulate values, for all alternatives: $v_{\boldsymbol{a}}^b = v\left(\boldsymbol{x}^b, \boldsymbol{a}\right), \quad b = 1,\ldots,B, \quad \boldsymbol{a} \in \boldsymbol{A}$
3. Simulate data: $\boldsymbol{y}^b \sim \left[\boldsymbol{y} \mid \boldsymbol{x}^b\right], \quad b = 1,\ldots,B$
4. Regress samples to fit conditional mean: $\hat{E}\left(v_a \mid \boldsymbol{y}\right)$

$$PoV(\boldsymbol{y}) \approx \frac{1}{B}\sum_{b=1}^{B} \max_{\boldsymbol{a}\in\boldsymbol{A}} \left\{ \hat{E}\left(v_{\boldsymbol{a}} \mid \boldsymbol{y}^b\right)\right\}$$

Regression results

$$\left(v_{\boldsymbol{a}}^{b}, \boldsymbol{y}^{b}\right), \qquad b = 1, \ldots, 1000, \qquad \boldsymbol{a} \in \boldsymbol{A}$$

We use partial least squares (PLS) regression, useful for the large-size seismic data sets.

	10-20 Hz seismic	**10-40 Hz seismic**	**10-60 Hz seismic**
VOI	578 (555-590)	583 (560-600)	585 (565-600)

VOI only very large, but slightly larger for more detail in the seismic data, since we are interested in 'overall' flow properties.

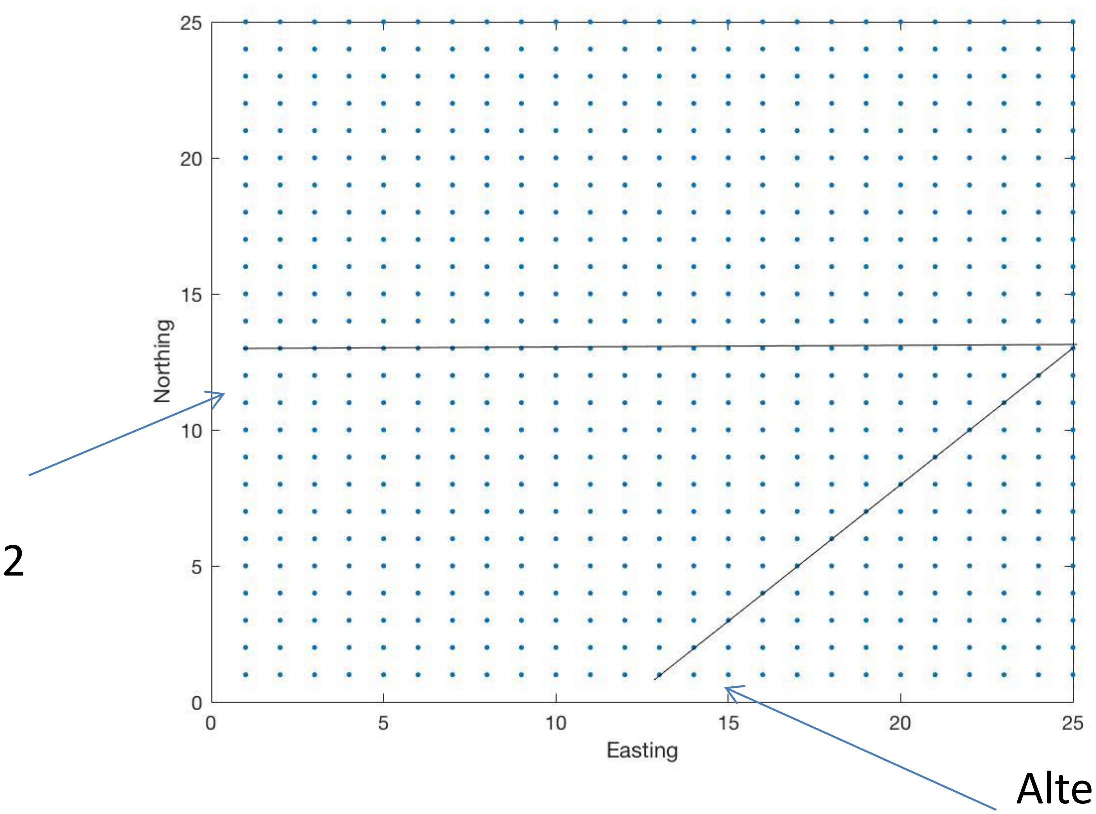
Wrap up

- The type of simulation and regression would be very case specific.
- If there are lots of alternatives, some kind of clustering of alternatives should be used.

- VOI approximation is difficult to check, but bootstrap (or bagging) can be used to study uncertainty, and to do sensitivity over different regression models.

Project case : alternatives

Suppose there is a 25 x 25 grid of reservoir variables. We want to flood the reservoir either from the west, or from the south.



Project case : models

There is uncertainty in the reservoir properties, possibly a channel with larger permeability in the middle, and some heterogeneity.

We can sample from the model as follows:

$$b = 1, \dots, B$$

- Draw a regression parameter:

$$\beta^b \sim p(\beta) = \text{Gamma}(1,1)$$

Use gamrnd or rgamma

- Draw a Gaussian process on the 25 x 25 grid:

$$\boldsymbol{x}_0^b \sim p\left(\boldsymbol{x}_0 \mid \beta^b\right) = N\left(-\beta^b \cdot \frac{(North-13)^2}{144}, \boldsymbol{\Sigma}\right)$$

Use Cholesky method

- Permeability is log-Gaussian:

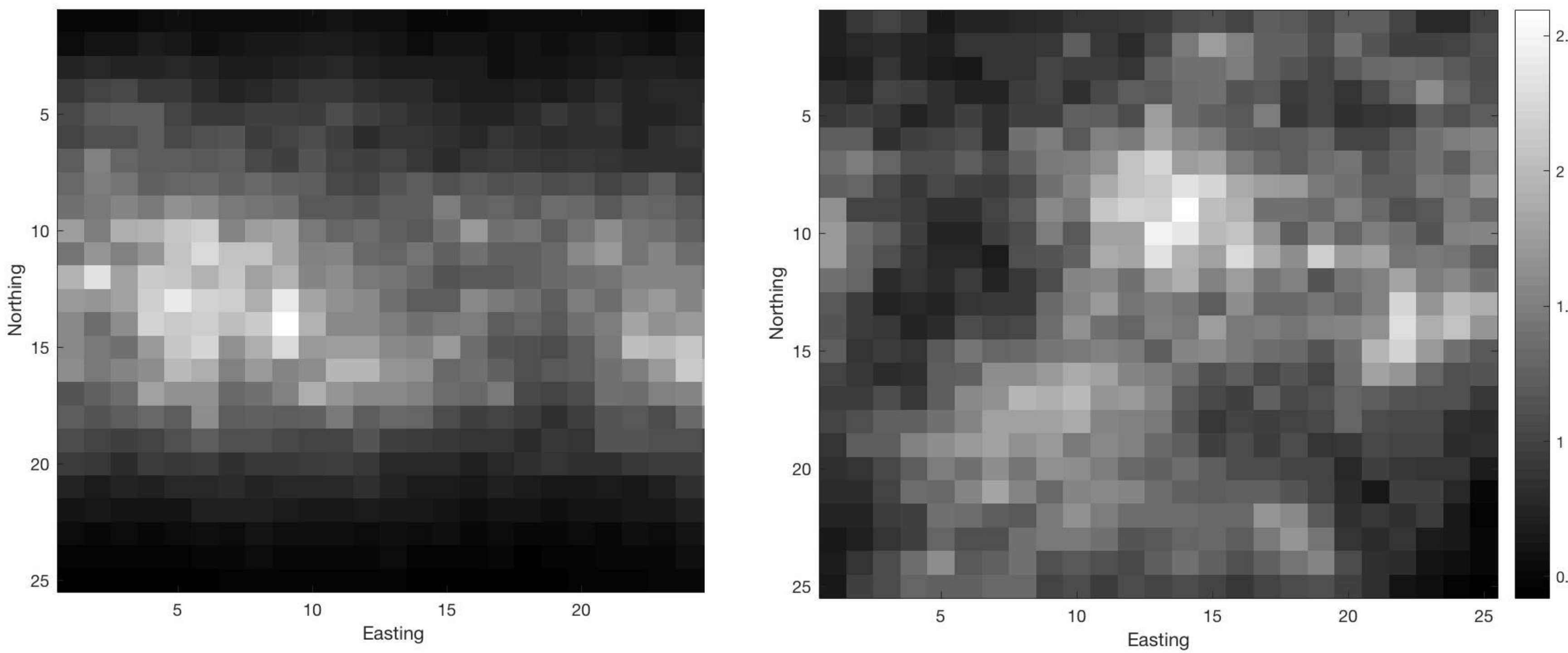
$$\boldsymbol{x}^b = \exp\left(\boldsymbol{x}_0^b\right)$$

Project case : models

$$\Sigma_{ij} = 0.3^2 \exp\left(-\frac{3\left|\mathbf{s}_i - \mathbf{s}_j\right|}{20} \right)$$

Variability due to the regression uncertainty and the spatial heterogeneity.

Two permeability realizations



Project case : values

Value is set as the time-of-flight: time it takes a particle to travel from the source to the sink. Smaller is better, larger 'value'. (This is used as a proxy for flow.)

For each alternative (west or south), we compute time-of-flight as follows:

$$v_a^b = \sum_{i \in l_a} d_i / x_i$$

$$b = 1, \ldots, B$$

Distance is 1 for 'west' alternative, 1.5 for 'south' alternative.

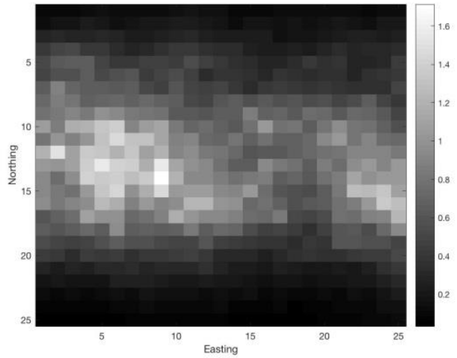
Sum inverse permeability variables along the line. Large permeability, smaller time of flight.

Project case : data

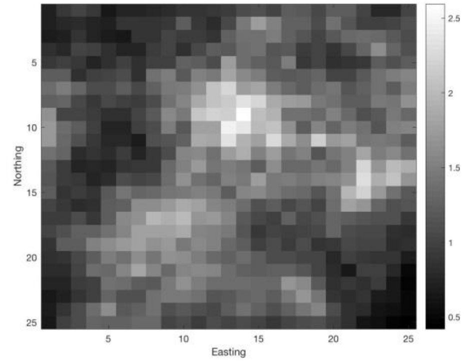
Data is the log-ratio of the variability in the center N-S line compared with the center E-W line. (This might be a result of processing seismic data.)

$$y^b = \log\left[\frac{\frac{1}{24}\sum_{i\in NS}\left(x_i^b - \overline{x}_{NS}^b\right)^2}{\frac{1}{24}\sum_{i\in EW}\left(x_i^b - \overline{x}_{EW}^b\right)^2}\right] \qquad b = 1,\ldots,B$$

Large data.



Small data.



Project case : regression

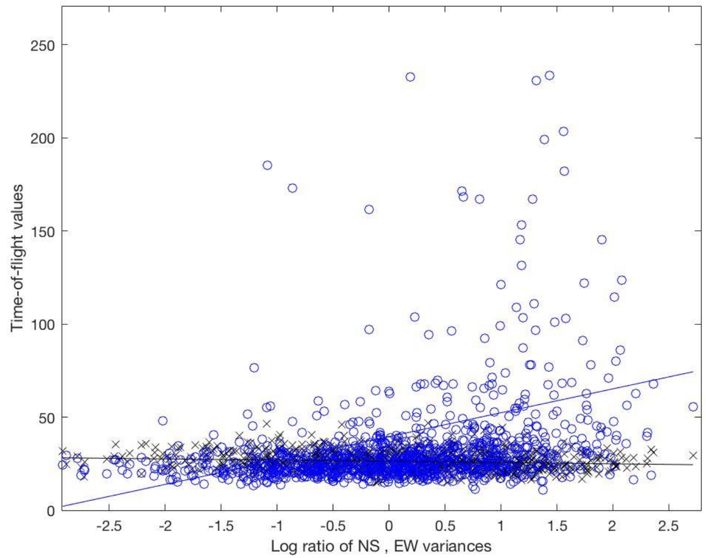
Estimate the conditional expected values by simple linear regression, using the samples of values and data. Do this for both ‘west’ and ‘south’ alternative.

$$\hat{E}\left(v_a \mid y\right) = \hat{\alpha}_{0,a} + \hat{\alpha}_{1,a} y \qquad \left(y^b, v_a^b\right), \quad b = 1, \ldots, B$$

$$PoV\left(\boldsymbol{y}\right) \approx \frac{1}{B} \sum_{b=1}^{B} \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ -\hat{E}\left(v_{\boldsymbol{a}} \mid \boldsymbol{y}^b\right) \right\}$$

$$PV \approx \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ -\frac{1}{B} \sum_{b=1}^{B} v_a^b \right\}$$

$$VOI \approx PoV\left(\boldsymbol{y}\right) - \mathrm{PV}$$



Project case : VOI approximation

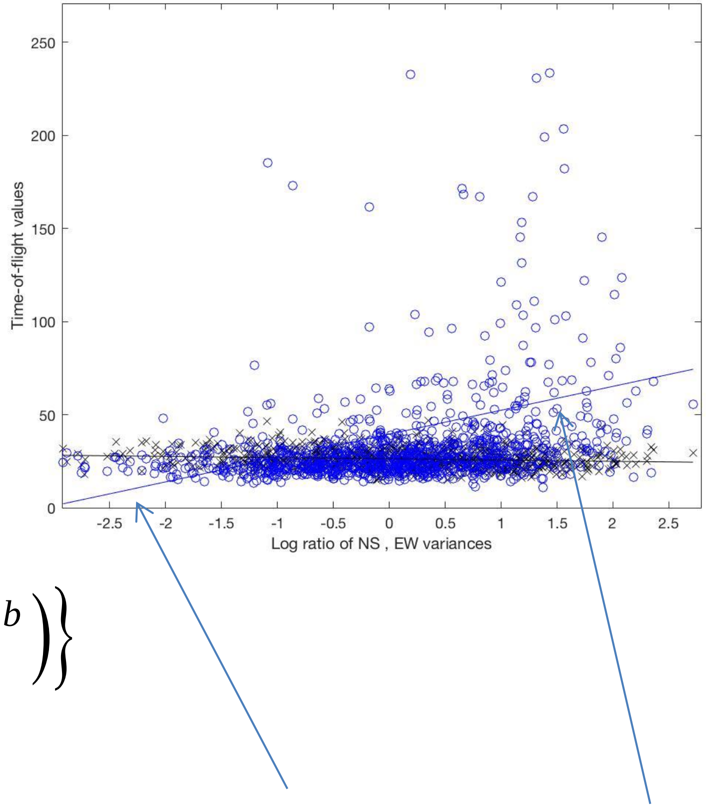
$$\left(y^b, v_a^b\right), \qquad b = 1, \ldots, B$$

$$PV \approx \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ -\frac{1}{B} \sum_{b=1}^{B} v_a^b \right\}$$

$$\hat{E}\left(v_a \mid y\right) = \hat{\alpha}_{0,a} + \hat{\alpha}_{1,a} y$$

$$PoV\left(\boldsymbol{y}\right) \approx \frac{1}{B} \sum_{b=1}^{B} \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ -\hat{E}\left(v_{\boldsymbol{a}} \mid \boldsymbol{y}^b\right) \right\}$$

$$VOI \approx PoV\left(\boldsymbol{y}\right) - \mathrm{PV}$$



Best alternative seems to depend on high or low data. This should give positive VOI.